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Some one-parameter families of commutative learning operators

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The first stochastic learning processes to be studied in any detail assumed that response probabilities are transformed linearly by path-independent stochastic operators. Such processes can also be viewed as infinite-state Markov chains. Since these initial studies were undertaken, three additional lines of investigation have appeared: finite-state Markov chains, path-dependent modifications of the linear operators, and path-independent nonlinear operators. This paper is concerned with an important class of nonlinear operators.

To abandon linearity may well be necessary, but to do so with abandon is pointless: no significant results can be proved about nonlinear stochastic processes unless some powerful constraint replaces linearity. So far, the only nonlinear operator model that has been studied in any detail is the so-called beta model (Bush, 1960; Bush, Galanter, and Luce, 1959; Kanal, 1960, 1962a, 1962b; Luce, 1959; Lamperti and Suppes, 1960). If we confine our attention to two-response experiments and let p_n denote the probability that response 1 occurs on trial n, then the beta-model operators are of the form

$$p_{n+1} = \frac{\beta p_n}{1 - p_n + \beta p_n},$$

where $\beta > 0$. Or, if we introduce the transformation $v_n = p_n/(1 - p_n)$, the operators become simply $v_{n+1} = \beta v_n$. In this form, it is easy to see that the operators commute with one another—i.e., that the order of application does not matter—which is one reason why the beta model is very nearly as tractable as the linear one.

This being so, it seems sensible to investigate the more general classes of one-parameter families of commutative-operator learning models. This we do, exhibiting a class of models whose members are all much like the beta model. We also examine which of the theorems now known about the beta model can be readily generalized to certain of these commutative-operator models.

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1. One-parameter families of learning operators

DEFINITION 1. *A one-parameter family of learning operators—or* **briefly a** *family of operators—is a triple* (X, A, F), *where* X *and* A *are (open or closed) intervals of real numbers and* F *is a function from* $X \times A$ *onto* X *for which*

i. *(closure) there is a function* $\theta : A \times A \to A$ *such that for all* $x \in X$ *and* $\alpha, \beta \in A$,

(1) $$F[F(x, \alpha), \beta] = F[x, \theta(\alpha, \beta)];$$

ii. *(associativity) for all* $x \in X$ *and* $\alpha, \beta, \gamma \in A$,

(2) $$F\{F[x, \theta(\alpha, \beta)], \gamma\} = F[F(x, \alpha), \theta(\beta, \gamma)];$$

iii. *(identity) there exists* $e \in A$ *such that for all* $x \in X$

(3) $$F(x, e) = x;$$

iv. *(continuity)* F *is continuous in each of its variables;*

and

v. *(uniqueness) for all* $x \in X$, $F(x, \alpha) = F(x, \alpha')$ *if and only if* $\alpha = \alpha'$.

When $X = U$, *the (open or closed) unit interval, the triple* (U, A, F) *is called* **a** family of stochastic operators.

The intended interpretation for stochastic operators is that if p_n is the probability of a particular response on trial n, then the corresponding probability on trial $n + 1$ is given by

$$p_{n+1} = F(p_n, \alpha),$$

where $\alpha \in A$ is a learning-rate parameter whose particular value is determined by the stimuli presented, the responses made, and the outcomes that resulted on certain previous trials.

The first condition of the definition says that an operator of the family applied to another operator of the family is a third operator of the same family. The second condition requires that the composition of any three operators shall be associative. Both of these conditions seem necessary if we are to consider compound operators members of the same family. Condition iii simply includes the identity operator as a member of the family. Condition iv—continuity—seems reasonable for learning operators. Condition v says that if two operators have identical effects on all $x \in X$, then they are exactly the same operator in the sense that the parameters are the same, and conversely. Note that condition v is equivalent to assuming that for each $x \in X$, F is strictly monotonic in the second variable.

THEOREM 1. *If (X, A, F) is a family of operators, then (A, A, θ) is a family of operators.*

PROOF. i and ii. By properties i and ii for (X, A, F),

$$\begin{aligned} F\{x, \theta[\theta(\alpha, \beta), \gamma]\} &= F\{F[x, \theta(\alpha, \beta)], \gamma\} \\ &= F\{F(x, \alpha), \theta(\beta, \gamma)\} \\ &= F\{x, \theta[\alpha, \theta(\beta, \gamma)]\}, \end{aligned}$$

and so by property v,

$$\theta[\theta(\alpha, \beta), \gamma] = \theta[\alpha, \theta(\beta, \gamma)],$$

which establishes both the closure and associativity of θ.

iii. By properties i and iii of (X, A, F),

$$F(x, \alpha) = F[F(x, \alpha), e] = F[x, \theta(\alpha, e)]$$

and

$$F(x, \alpha) = F[F(x, e), \alpha] = F[x, \theta(e, \alpha)],$$

so by property v,

$$\theta(\alpha, e) = \theta(e, \alpha) = \alpha.$$

iv. A detailed $\varepsilon - \delta$ proof that θ is continuous in the first variable can be based upon the observation that because F is strictly monotonic in the second variable, $\theta(\alpha, \beta) - \theta(\alpha', \beta)$ is small provided that

$$F[x, \theta(\alpha, \beta)] - F[x, \theta(\alpha', \beta)] = F[F(x, \alpha), \beta] - F[F(x, \alpha'), \beta]$$

is small, which it is by the continuity of F provided that $F(x, \alpha) - F(x, \alpha')$ is small, which it is when $\alpha - \alpha'$ is small. A similar argument shows that θ is continuous in the second variable.

v. If $\theta(\alpha, \beta) = \theta(\alpha, \beta')$ for all $\alpha \in A$, then for all $x \in X$,

$$F[x, \theta(\alpha, \beta)] = F[x, \theta(\alpha, \beta')],$$

which by property *i* implies that

$$F[F(x, \alpha), \beta] = F[F(x, \alpha), \beta'].$$

Because F is onto X, property v implies that $\beta = \beta'$. ||

THEOREM 2. *Let (X, A, F) be a family of operators, let f be a continuous, strictly monotonic-increasing function from X onto Y, let g be a continuous, strictly monotonic-increasing function from A onto B, and let $G : Y \times B \rightarrow Y$ be defined by*

$$G(y, \beta) = f\{F[f^{-1}(y), g^{-1}(\beta)]\}.$$

Then (Y, B, G) is a family of operators.

PROOF. i. Let θ_F be the function of Eq. (1) for (X, A, F). Then a routine check shows that

$$\theta_G(\beta, \gamma) = g\{\theta_F[g^{-1}(\beta), g^{-1}(\gamma)]\}$$

is the corresponding function for (Y, B, G).

ii. By the associativity of θ_F (Theorem 1),

$$\begin{aligned} \theta_G[\theta_G(\beta, \gamma), \delta] &= g(\theta_F\{g^{-1}[\theta_G(\beta, \gamma)], g^{-1}(\delta)\}) \\ &= g(\theta_F\{\theta_F[g^{-1}(\beta), g^{-1}(\gamma)], g^{-1}(\delta)\}) \\ &= g(\theta_F\{g^{-1}(\beta), \theta_F[g^{-1}(\gamma), g^{-1}(\delta)]\}) \\ &= g(\theta_F\{g^{-1}(\beta), g^{-1}[\theta_G(\gamma, \delta)]\}) \\ &= \theta_G[\beta, \theta_G(\gamma, \delta)]. \end{aligned}$$

This result immediately yields Eq. (2) for (Y, B, G).

iii. For any $y \in Y$,

$$\begin{aligned} G[y, g(e)] &= f(F\{f^{-1}(y), g^{-1}[g(e)]\}) \\ &= f\{F[f^{-1}(y), e]\} \\ &= f[f^{-1}(y)] \\ &= y, \end{aligned}$$

which shows that $g(e)$ satisfies Eq. (3).

iv. A routine check using the continuity of F, f, and g implies the continuity of G.

v. If $G(y, \beta) = G(y, \beta')$ for all $y \in Y$, then because f^{-1} exists,

$$F[f^{-1}(y), g^{-1}(\beta)] = F[f^{-1}(y), g^{-1}(\beta')].$$

By property v of Def. 1, $g^{-1}(\beta) = g^{-1}(\beta')$, so $\beta = \beta'$. ||

2. Commutative families of operators

DEFINITION 2. *A family of operators (X, A, F) is* commutative *if for all $x \in X$ and all $\alpha, \beta \in A$,*

$$F[F(x, \alpha), \beta] = F[F(x, \beta), \alpha]. \tag{4}$$

THEOREM 3. *Let (X, A, F) be a family of operators. Then (X, A, F) is commutative if and only if (A, A, θ) is commutative.*

PROOF. Because (X, A, F) is a family of operators, so is (A, A, θ) by Theorem 1, and $\theta[\theta(\alpha, \beta), \gamma] = \theta[\alpha, \theta(\beta, \gamma)]$. Suppose that (X, A, F) is commutative. Then by Eqs. (1) and (4), for all $x \in X$ and $\alpha, \beta \in A$,

$$\begin{aligned} F[x, \theta(\alpha, \beta)] &= F[F(x, \alpha), \beta] \\ &= F[F(x, \beta), \alpha] \\ &= F[x, \theta(\beta, \alpha)]. \end{aligned}$$

So by property v,

$$\theta(\alpha, \beta) = \theta(\beta, \alpha).$$

Conversely, if the last equation holds, then

$$\begin{aligned} F[F(x, \alpha), \beta] &= F[x, \theta(\alpha, \beta)] \\ &= F[x, \theta(\beta, \alpha)] \\ &= F[F(x, \beta), \alpha], \end{aligned}$$

and so (X, A, F) is commutative. Thus, we have shown that $\theta(\alpha, \beta) = \theta(\beta, \alpha)$, for all $\alpha, \beta \in A$, if and only if (X, A, F) is commutative.

By the first part of Theorem 1, we know that the θ_θ of (A, A, θ) is simply θ, so (X, A, F) is commutative if and only if $\theta(\alpha, \beta) = \theta(\beta, \alpha)$, which is the same as $\theta_\theta(\alpha, \beta) = \theta_\theta(\beta, \alpha)$, and this holds if and only if (A, A, θ) is commutative. ||

Theorem 4. *Under the conditions of Theorem 2, (X, A, F) is commutative if and only if (Y, B, G) is commutative.*

Proof. Using the first part of the proof of Theorem 3 and the representation of θ_G in terms of θ_F given in the proof of Theorem 2, we see that (X, A, F) is commutative if and only if, for all $\alpha, \beta \in A$,

$$\theta_F[g^{-1}(\alpha), g^{-1}(\beta)] = \theta_F[g^{-1}(\beta), g^{-1}(\alpha)],$$

which is equivalent to

$$\begin{aligned}\theta_G(\alpha, \beta) &= g\{\theta_F[g^{-1}(\alpha), g^{-1}(\beta)]\}\\ &= g\{\theta_F[g^{-1}(\beta), g^{-1}(\alpha)]\}\\ &= \theta_G(\beta, \alpha),\end{aligned}$$

which in turn is equivalent to the commutativity of (Y, B, G). ||

3. Strictly montonic-increasing families of operators

Definition 3. *A family of operators (X, A, F) is* strictly monotonic increasing *if for all $x \in X$ and $\alpha, \alpha' \in A$ such that $\alpha < \alpha'$, $F(x, \alpha) < F(x, \alpha')$, and if for all $\alpha \in A$ and $x, x' \in X$ such that $x < x'$, $F(x, \alpha) < F(x', \alpha)$.*

Theorem 5. *If (X, A, F) is a strictly monotonic increasing family of operators, then (A, A, θ) is a strictly monotonic-increasing family of operators. Moreover, there is a positive, continuous, strictly monotonic function g defined over A such that*

$$\theta(\alpha, \beta) = g^{-1}[g(\alpha)g(\beta)], \tag{5}$$

and both families are commutative.

Proof. By Theorem 1 we know that (A, A, θ) is a family of operators. To show that (A, A, θ) is strictly monotonic-increasing, suppose that $\beta < \beta'$. Then by the monotonicity of F we have

$$\begin{aligned}F[x, \theta(\alpha, \beta)] &= F[F(x, \alpha), \beta]\\ &< F[F(x, \alpha), \beta']\\ &= F[x, \theta(\alpha, \beta')],\end{aligned}$$

and so by the monotonicity of F, $\theta(\alpha, \beta) < \theta(\alpha, \beta')$. A similar proof shows that $\alpha < \alpha'$ implies that $\theta(\alpha, \beta) < \theta(\alpha', \beta)$.

Given that (A, A, θ) is a strictly monotonic-increasing family of operators,

Aczél (1948) has proved the rather deep result that there exists a continuous strictly monotonic function g^* such that

$$\theta(\alpha, \beta) = g^{*-1}[g^*(\alpha) + g^*(\beta)].$$

Setting $g = \exp g^*$, we obtain Eq. (5).

It is evident from Eq. (5) that $\theta(\alpha, \beta) = \theta(\beta, \alpha)$ and so by the argument used in Theorem 3, (A, A, θ) is commutative. From this and Eq. (1) it follows easily that (X, A, F) is also commutative. ||

Examples of Eq. (5)

1. If $\theta(\alpha, \beta) = \alpha\beta$, then $g(\alpha\beta) = g(\alpha)g(\beta)$, which with the continuity of g implies that $g(\alpha) = \alpha^k$, for some positive constant k.
2. If $\theta(\alpha, \beta) = \alpha + \beta$, then similarly $g(\alpha) = \exp k\alpha$, for some positive constant k.

Theorem 6. *Let (X, A, F) be a family of strictly monotonic-increasing operators with the following property: there exists an $x^* \in X$ and a continuous, strictly monotonic-increasing function φ from X onto A such that for every $x \in X$, $F[x^*, \varphi(x)] = x$. Then there exist positive, continuous, strictly monotonic-increasing functions f on X and g on A such that for all $x \in X$ and $\alpha \in A$,*

$$F(x, \alpha) = f^{-1}[f(x)g(\alpha)], \tag{6}$$

and g satisfies Eq. (5).

Proof. The following proof is for the most part adapted from Aczél (1961, p. 189).

For $x, y \in X$, define

$$G(x, y) = F[x, \varphi(y)].$$

We show that (X, X, G) is a strictly monotonic-increasing family of operators.

i. G is closed because F is.

ii. Because (X, A, F) is commutative (Theorem 5),

$$\begin{aligned}G(x, y) &= F[x, \varphi(y)]\\ &= F\{F[x^*, \varphi(x)], \varphi(y)\}\\ &= F\{F[x^*, \varphi(y)], \varphi(x)\}\\ &= F(y, \varphi(x)]\\ &= G(y, x).\end{aligned}$$

Using this result, we now show that G is associative:

$$\begin{aligned}G[G(x, y), z] &= G[G(y, x), z]\\ &= F\{F[y, \varphi(x)], \varphi(z)\}\\ &= F\{F[y, \varphi(z)], \varphi(x)\}\\ &= G[G(y, z), x]\\ &= G[x, G(y, z)].\end{aligned}$$

iii. Because $F(x^*, e) = x^*$, $\varphi(x^*) = e$. Thus,

$$\begin{aligned} G(x^*, x) &= G(x, x^*) \\ &= F[x, \varphi(x^*)] \\ &= F(x, e) \\ &= x, \end{aligned}$$

so x^* is the identity of (X, X, G).

iv. Because F and φ are continuous, so is G.

v. If $G(x, y) = G(x, y')$ for all $x \epsilon X$, then $F[x, \varphi(y)] = F[x, \varphi(y')]$ and so by property v of (X, A, F), $\varphi(y) = \varphi(y')$. By the strict monotonicity of φ, $y = y'$.

Suppose that $y < y'$. Then because φ is strictly increasing, $\varphi(y) < \varphi(y')$. From this inequality and the fact that F is strictly increasing,

$$\begin{aligned} G(x, y) &= F[x, \varphi(y)] \\ &< F[x, \varphi(y')] \\ &= G(x, y'), \end{aligned}$$

so (X, X, G) is a strictly monotonic-increasing family of operators.

By the Aczél theorem previously used in Theorem 5, there exists a positive, continuous, strictly monotonic-increasing function f such that

$$G(x, y) = f^{-1}[f(x)f(y)];$$

hence

$$\begin{aligned} F(x, \alpha) &= G[x, \varphi^{-1}(\alpha)] \\ &= f^{-1}[f(x)g(\alpha)], \end{aligned}$$

where $g = f(\varphi^{-1}) > 0$.

To show that g satisfies Eq. (5), substitute Eq. (6) into Eq. (1). ||

Suppose that (U, A, F) is a family of stochastic operators that satisfies the conditions of Theorem 6. Then if we let f define a new variable $y_n = f(p_n) > 0$, and let g define a new parameter $\beta = g(\alpha) > 0$, Eq. (6) states that

$$\begin{aligned} y_{n+1} &= f(p_{n+1}) \\ &= f[F(p_n, \alpha)] \\ &= f(p_n)g(\alpha) \\ &= y_n\beta. \end{aligned}$$

Thus, in terms of this transformed variable, such a family of commutative (Theorem 5) stochastic operators takes on the simple multiplicative form of the beta model.

The beta model is, of course, an example of this theorem, where $f(p) = p/(1-p)$ and $g(\alpha) = \alpha$.

Note that the special property needed to prove Theorem 6 is met if $Y = B$ and $1 \epsilon Y$. To show this, define $x^* = f^{-1}(1)$ and $\varphi(x) = g^{-1}[f(x)]$. Then

$$\begin{aligned} F[x^*, \varphi(x)] &= f^{-1}\{f(x^*)g[\varphi(x)]\} \\ &= f^{-1}[1 \cdot f(x)] \\ &= x. \end{aligned}$$

4. Quasi-multiplicative families of operators

Not all families of commutative operators are strictly monotonic-increasing. For example, the family of linear operators with a fixed limit point $\lambda \epsilon U$, defined by

$$F(p, \alpha) = \alpha p + (1-\alpha)\lambda \qquad (-1 \leq \alpha \leq 1),$$

fails to satisfy Theorem 6 in spite of the fact that if we let $f(p) = p - \lambda$ and $g(\alpha) = \alpha$, then we can write F as

$$F(p, \alpha) = f^{-1}[f(p)g(\alpha)].$$

Theorem 6 does not apply because the function F is not strictly monotonic-increasing in the first variable; this shows up in the fact that f is not positive for all $p \epsilon U$.

So far as I know, no representation theorem similar to Theorem 6 has been proved for commutative operators when the strict monotonicity condition is dropped. The only result I know of is the following sufficient condition.

Theorem 7. *Let X, Y, A, and B be real intervals such that*

i. *there are continuous, strictly monotonic-increasing functions f from X onto Y and g from A onto B;*

ii. *if $y \epsilon Y$ and $\beta \epsilon B$, then $y\beta \epsilon Y$;*

iii. *if $\beta, \gamma \epsilon B$, then $\beta\gamma \epsilon B$; and*

iv. *$1 \epsilon B$.*

If for $x \epsilon X$ and $\alpha \epsilon A$, F is defined by

$$F(x, \alpha) = f^{-1}[f(x)g(\alpha)], \tag{7}$$

then (X, A, F) is a family of commutative operators with $e = g^{-1}(1)$ and

$$\theta(\alpha, \beta) = g^{-1}[g(\alpha)g(\beta)]. \tag{8}$$

Proof. Define G as follows:

$$G(y, \beta) = y\beta \qquad (y \epsilon Y; \beta \epsilon B).$$

Using hypotheses ii, iii, and iv, it is easy to see that (Y, B, G) is a family of commutative operators. Because

$$\begin{aligned} F(x, \alpha) &= f^{-1}[f(x)g(\alpha)] \\ &= f^{-1}\{G[f(x), g(\alpha)]\}, \end{aligned}$$

Theorems 2 and 4 imply that (X, A, F) is also a family of commutative operators. Equation (8) follows from Eqs. (7) and (1). ||

Because of the form of Eq. (7), we make the following definition.

Definition 4. *Any family of operators* (X, A, F) *for which there are intervals* Y *and* B *and functions* f *and* g *that satisfy conditions i–iv and Eq.* (7) *of Theorem* 7 *is called* quasi-multiplicative.

This class of families includes not only the strictly monotonic-increasing families of Theorem 6, but also the linear operators with a fixed limit point.

It should be noted that for quasi-multiplicative families we can always choose our parameters so that $\theta(\alpha, \beta) = \alpha\beta$. This follows from Theorems 2 and 4 and Eq. (8) if we replace the given quasi-multiplicative family (X, A, F) by (X, B, F^*), where

$$F^*(x, \beta) = F[x, g^{-1}(\beta)].$$

The beta model, which is an example of a strictly monotonic-increasing family, and the linear model with a fixed limit point, which is not, illustrate two inherently different classes of quasi-multiplicative operators. There are those models such as the linear model, in which the range of f is a bounded interval, and those such as the beta model, in which it is unbounded. In the first case, it is easy to see from hypothesis ii of Theorem 7 that the range of g, B, is also bounded; in fact, if $\beta \in B$, then $|\beta| \leq 1$. Care must be taken to distinguish between these two cases, for a general theorem usually applies only to one of them.

5. Complete families of stochastic operators[1]

Thus far we have not considered any condition that restricts our attention to families of stochastic operators, i.e., to families where $X = U =$ the unit interval. We now consider these families.

In any two-response experiment, we need only study how one response probability changes over trials—the other probability is automatically the complement. Because we are free to state the learning model in terms of either response probability, it seems reasonable to demand that the form of the model should not depend upon that arbitrary choice. Thus, if a particular one-parameter family of stochastic operators describes the learning process when it is stated in terms of one response, then the same family applied to the other response probability, quite possibly with different parameter values, should equally well describe the process. We capture this requirement in the following definition.

Definition 5. *A family of stochastic operators* (U, A, F) *is* complete *if for each* $\alpha \in A$ *there exists a unique* $\psi(\alpha) \in A$ *such that for all* $p \in U$,

(9) $$F(p, \alpha) + F[1 - p, \psi(\alpha)] = 1.$$

It is easy to see that the beta-model family is complete with $\psi(\alpha) = 1/\alpha$.

Not all families of commutative operators are complete. For example, let $f(p) = \log p$, let g be any continuous strictly monotonic-increasing function, and define F by Eq. (7). Then $F(p, \alpha) = p^{g(\alpha)}$. Except for $g \equiv 1$, it is clear that we cannot choose ψ such that for all $p \in U$,

$$p^{g(\alpha)} + (1 - p)^{g[\psi(\alpha)]} = 1.$$

Some care must be exercised in interpreting the condition of completeness when the family depends upon additional fixed parameters, as do the commutative linear operators

$$F(p, \alpha, \lambda) = \alpha p + (1 - \alpha)\lambda,$$

where $\lambda \in U$ is fixed, and we have extended the notation to cover two-parameter families of operators. Observe that when $\lambda \neq \frac{1}{2}$, we cannot choose ψ such that for all p,

$$F(p, \alpha, \lambda) + F[1 - p, \psi(\alpha), \lambda] = 1.$$

The difficulty is clear: λ is the limit point of the operators, and so the complementary process must have the limit point $1 - \lambda$. However, there is no objection to associating different fixed parameters with different responses, in which case it is easy to see that for all $p \in U$,

$$F(p, \alpha, \lambda) + F(1 - p, \alpha, 1 - \lambda) = 1,$$

and so the full family of linear operators is complete and for each λ it is commutative.

The following two theorems do not apply to families (e.g. the linear operators) where a different fixed constant is chosen for each response.

Theorem 8. *If* (U, A, F) *is a complete family of stochastic operators and if* θ *is defined by Eq.* (1) *and* ψ *by Eq.* (9), *then*

i. *for all* $\alpha, \beta \in A$, $\psi[\theta(\alpha, \beta)] = \theta[\psi(\alpha), \psi(\beta)]$;

ii. ψ *is continuous; and*

iii. *for all* $\alpha \in A$, $\psi[\psi(\alpha)] = \alpha$.

Proof. i. Using completeness three times and Eq. (1) twice, we obtain

$$\begin{aligned} F\{1 - p, \psi[\theta(\alpha, \beta)]\} &= 1 - F[p, \theta(\alpha, \beta)] \\ &= 1 - F[F(p, \alpha), \beta] \\ &= F\{1 - F(p, \alpha), \psi(\beta)\} \\ &= F\{F[1 - p, \psi(\alpha)], \psi(\beta)\} \\ &= F\{1 - p, \theta[\psi(\alpha), \psi(\beta)]\}, \end{aligned}$$

and so by property v of Def. 1 the first assertion follows.

ii. If $\alpha - \alpha'$ is small, then by the continuity of F, so is $F(p, \alpha) - F(p, \alpha')$. By completeness,

$$F(p, \alpha) - F(p, \alpha') = F[1 - p, \psi(\alpha')] - F[1 - p, \psi(\alpha)].$$

Because this quantity is small and F is continuous and strictly monotonic in the second variable, it follows that $\psi(\alpha') - \psi(\alpha)$ is small, establishing the continuity of ψ.

[1] I wish to thank Dr. Saul Sternberg for bringing to my attention the problem discussed in this section.

iii. Using completeness twice, we obtain

$$F(p, \alpha) = 1 - F[1 - p, \psi(\alpha)] = F\{p, \psi[\psi(\alpha)]\},$$

and so $\psi[\psi(\alpha)] = \alpha$ by property v of Def. 1. ||

COROLLARY. *If* $\theta(\alpha, \beta) = \alpha\beta$, *then* $\psi(\alpha) = \alpha^k$, *where* $k = 1$ *or* -1.

PROOF. From $\theta(\alpha, \beta) = \alpha\beta$ and assertion i of Theorem 8, $\psi(\alpha\beta) = \psi(\alpha)\psi(\beta)$. It is well known that the continuous solutions to this equation are $\psi(\alpha) = \alpha^k$, for some constant k. From assertion iii, $k^2 = 1$. ||

THEOREM 9. *Let* (U, A, F) *be a quasi-multiplicative family of stochastic operators such that for each pair* $p, q \in U$ *there exists an* $\alpha(p, q) \in A$ *for which either* $F[p, \alpha(p, q)] = q$ *or* $F[q, \alpha(p, q)] = p$. *Such a family is complete if and only if either*

$$f(p)f(1 - p) = f(\tfrac{1}{2})^2 > 0, \text{ for all } p \in U, \tag{10}$$

and

$$\beta \in B \text{ implies that } \frac{1}{\beta} \in B, \tag{11}$$

or

$$f(p) + f(1 - p) = 0, \text{ for all } p \in U, \tag{12}$$

In the first case, $g[\psi(\alpha)] = 1/g(\alpha)$ *and in the second,* $g[\psi(\alpha)] = g(\alpha)$.

PROOF. We establish the necessity first. Two cases must be distinguished. First, suppose that $f(\frac{1}{2}) \neq 0$. For a fixed $\alpha \in A$, let

$$h(p) = f(p)g(\alpha) - f(1 - p).$$

Because f is continuous, so is h. Because f is strictly increasing and because $f(p)g(\alpha) \in Y$,

$$f(0) \leq f(p)g(\alpha) \leq f(1),$$

where $f(0) = \lim_{p\to 0} f(p)$ and $f(1) = \lim_{p\to 1} f(p)$. Thus,

$$h(0) = f(0)g(\alpha) - f(1) \leq 0 \leq f(1)g(\alpha) - f(0) = h(1),$$

which together with the continuity of h means that there exists a $p(\alpha) \in U$ such that $h[p(\alpha)] = 0$, i.e., such that $f[p(\alpha)]g(\alpha) = f[1 - p(\alpha)]$. Note that $f[p(\alpha)] \neq 0$, because otherwise, by what we have just shown,

$$f[1 - p(\alpha)] = f[p(\alpha)]g(\alpha) = 0.$$

This, together with the strict monotonicity of f, implies that $p(\alpha) = \frac{1}{2}$, contradicting the assumption that $f(\frac{1}{2}) \neq 0$.

From completeness and Eq. (7), we know that for the above α and its $p(\alpha)$ there exists $\psi(\alpha) \in A$ such that

$$\begin{aligned} f^{-1}\{f[1 - p(\alpha)]g[\psi(\alpha)]\} &= F[1 - p(\alpha), \psi(\alpha)] \\ &= 1 - F[p(\alpha), \alpha] \\ &= 1 - f^{-1}\{f[p(\alpha)]g(\alpha)\} \\ &= 1 - f^{-1}\{f[1 - p(\alpha)]\} \\ &= p(\alpha). \end{aligned}$$

Therefore,

$$\begin{aligned} f[p(\alpha)] &= f[1 - p(\alpha)]g[\psi(\alpha)] \\ &= f[p(\alpha)]g(\alpha)g[\psi(\alpha)]. \end{aligned}$$

Dividing by $f[p(\alpha)] \neq 0$, we obtain

$$g[\psi(\alpha)] = 1/g(\alpha),$$

which establishes Eq. (11).

For any $p \in U$, we know by hypothesis that there exists an $\alpha(p) \in A$ such that either $F[p, \alpha(p)] = \frac{1}{2}$ or $F[\frac{1}{2}, \alpha(p)] = p$. Actually, both hold. For example, if the second is true, then by Eq. (7), $f(p) = f(\frac{1}{2})g[\alpha(p)]$. By Eq. (11), $1/g[\alpha(p)] = g[\alpha'(p)] \in B$, so $f(\frac{1}{2}) = f(p)/g[\alpha(p)] = f(p)g[\alpha'(p)]$, which is equivalent to $F[p, \alpha'(p)] = \frac{1}{2}$. From this and completeness,

$$F\{1 - p, \psi[\alpha(p)]\} = 1 - F[p, \alpha(p)] = \tfrac{1}{2},$$

and so

$$\begin{aligned} f(p)g[\alpha(p)] &= f(\tfrac{1}{2}) \\ &= f(1 - p)g\{\psi[\alpha(p)]\} \\ &= f(1 - p)/g[\alpha(p)]. \end{aligned}$$

Eliminating $g[\alpha(p)]$, we obtain Eq. (10).

Next, suppose that $f(\frac{1}{2}) = 0$. As we noted earlier, if (U, A, F) is quasi-multiplicative, so then is (U, B, F^*), where $F^*(p, \beta) = F[p, g^{-1}(\beta)]$. Moreover, if (U, A, F) is complete, so is (U, B, F^*) with $\psi^*(\beta) = g\{\psi[g^{-1}(\beta)]\}$:

$$\begin{aligned} F^*(p, \beta) + F^*[1 - p, \psi^*(\beta)] &= F[p, g^{-1}(\beta)] + F\{1 - p, \psi[g^{-1}(\beta)]\} \\ &= 1. \end{aligned}$$

In (U, B, F^*), we have $\theta^*[g(\alpha), g(\beta)] = g(\alpha)g(\beta)$, and so by the corollary to Theorem 8, we know that $\psi^*[g(\alpha)] = g(\alpha)^k$, where $k = 1$ or -1. For $p \neq \frac{1}{2}$, by hypothesis there exists $\alpha(p)$ such that either $f(p)g[\alpha(p)] = f(\frac{1}{2}) = 0$ or $f(\frac{1}{2})g[\alpha(p)] = 0 = f(p) \neq 0$. As the latter is clearly impossible, the former must hold, and so $g[\alpha(p)] = 0$. If $k = -1$, then $\psi^*\{g[\alpha(p)]\} = 1/g[\alpha(p)]$ does not exist, contrary to the assumption of completeness. Thus, $k = 1$, in which case $\psi(\alpha) = g^{-1}\{\psi^*[g(\alpha)]\} = g^{-1}[g(\alpha)] = \alpha$.

Let $p, q \in U$, where $p, q \neq \frac{1}{2}$. Then by the strict monotonicity of f and the fact that $f(\frac{1}{2}) = 0$, we see that $f(p)$, $f(q)$, $f(1 - p)$, and $f(1 - q) \neq 0$. By hypothesis, there exists an $\alpha \in A$ such that either $f(q) = f(p)g(\alpha)$ or $f(p) = f(q)g(\alpha)$; so by completeness, either

$$f(1 - q) = f(1 - p)g[\psi(\alpha)] = f(1 - p)g(\alpha),$$

or

$$f(1 - p) = f(1 - q)g(\alpha).$$

In either case, eliminating $g(\alpha)$, we obtain

$$\frac{f(p)}{f(1 - p)} = \frac{f(q)}{f(1 - q)},$$

that is, for $p \neq \frac{1}{2}$, $f(p)/f(1-p) = K$, a constant. Thus,

$$1 = \frac{f(p)}{f(1-p)} \cdot \frac{f(1-p)}{f(p)} = K^2.$$

Because f is strictly monotonic, $f(\frac{1}{2}) = 0$, and either $p < \frac{1}{2}$ and $1 - p > \frac{1}{2}$ or $p > \frac{1}{2}$ and $1 - p < \frac{1}{2}$, it follows that $f(p)/f(1-p)$ is negative, so $K = -1$ and $f(p) + f(1-p) = 0$. For $p = \frac{1}{2}$, this equation is also met, thus proving Eq. (12).

Conversely, suppose that Eqs. (10) and (11) hold. Given any $\alpha \epsilon A$, choose $\psi(\alpha) \epsilon A$ such that $g[\psi(\alpha)] = 1/g(\alpha)$, which is possible by Eq. (11), and let $q(p) = f^{-1}[f(p)g(\alpha)]$. By Eq. (10),

$$\begin{aligned} f^{-1}\{f(1-p)g[\psi(\alpha)]\} &= f^{-1}[f(\tfrac{1}{2})^2/f(p)g(\alpha)] \\ &= f^{-1}\{f(\tfrac{1}{2})^2/f[q(p)]\} \\ &= f^{-1}\{f[1-q(p)]\} \\ &= 1 - q(p). \end{aligned}$$

Thus,

$$\begin{aligned} F(p, \alpha) + F[1-p, \psi(\alpha)] &= f^{-1}[f(p)g(\alpha)] + f^{-1}\{f(1-p)g[\psi(\alpha)]\} \\ &= q(p) + 1 - q(p) \\ &= 1, \end{aligned}$$

and so the family is complete.

Next, suppose that Eq. (12) holds. For any $\alpha \epsilon A$, let $\psi(\alpha) = \alpha$ and let $q(p) = f^{-1}[f(p)g(\alpha)]$. Then

$$\begin{aligned} f[1-q(p)] &= -f[q(p)] \\ &= -f(p)g(\alpha) \\ &= f(1-p)g[\psi(\alpha)]. \end{aligned}$$

Thus,

$$\begin{aligned} f^{-1}[f(p)g(\alpha)] + f^{-1}\{f(1-p)g[\psi(\alpha)]\} &= q(p) + 1 - q(p) \\ &= 1, \end{aligned}$$

and so the family is complete. ||

Several comments should be made about Theorem 9. First, the sufficiency proofs use only the assumption that the operators are quasi-multiplicative, whereas the necessity requires the extra hypothesis that for any p and q there is an α such that either $F(p, \alpha) = q$ or $F(q, \alpha) = p$. Some insight into the role of this hypothesis can be obtained by defining pRq if there exists an $\alpha \epsilon A$ such that $q = F(p, \alpha)$. By Eq. (3), R is a reflexive relation, and by a simple application of Eq. (1) it is transitive. Moreover, if every operator has an inverse in the sense that for each $\alpha \epsilon A$ there exists an $\alpha^{-1} \epsilon A$ such that for all $p \epsilon U$, $F[F(p, \alpha), \alpha^{-1}] = p$, then it is easy to see that R is symmetric. Thus, it is an equivalence relation, and the extra hypothesis implies that there is only one equivalence class. Without this hypothesis, there may be several equivalence classes, and transitions can occur only between members of the same equivalence class.

Second, if f is a positive function that satisfies Eq. (10), then $f^*(p) = \log [f(p)/f(\frac{1}{2})]$ satisfies Eq. (12), and conversely.

Third, the class of functions satisfying Eq. (10) is not small. Let h be any continuous, strictly monotonic-increasing function from the half-open interval $(0, \frac{1}{2}]$ onto $(0, h(\frac{1}{2})]$, where $0 < h(\frac{1}{2}) < 1$. If we define f as follows:

$$f(p) = \begin{cases} h(p) & \text{if } p \leq \frac{1}{2}, \\ h(\frac{1}{2})^2/h(1-p) & \text{if } p \geq \frac{1}{2}, \end{cases}$$

then it is easy to check that Eq. (10) is met.

Fourth, the beta model satisfies the first condition and the linear model with $\lambda = \frac{1}{2}$ satisfies the second one.

Corollary 1. *Let* (U, A, F) *be a complete, quasi-multiplicative family of stochastic operators that satisfies the hypothesis of Theorem* 9. *If Eqs.* (10) *and* (11) *are met, then* U *is open, the range of* g, B, *is the positive reals, the range of* f, Y, *is either the positive or the negative reals, and every operator has an inverse.*

Proof. Eq. (10) implies that $f(p)$ and $f(1-p)$ are both non-zero and have the same sign. If $f(\frac{1}{2}) > 0$, then the fact that f is monotonic increasing implies that they are both positive, whereas if $f(\frac{1}{2}) < 0$, it implies that they are both negative. Thus, Y is a subset either of the positive or of the negative reals.

Next, we show that $B = \{y/z \mid y, z \epsilon Y\}$. By the hypothesis of Theorem 9 and Eq. (11), we see that $\{y/z \mid y, z \epsilon Y\} \subseteq B$. By the first part of the proof of Theorem 9, $B \subseteq \{f(1-p)/f(p) \mid p \epsilon U\} \subseteq \{y/z \mid y, z \epsilon Y\}$, thus proving the assertion. Because Y is a subset of either the positive or the negative reals, B must be a subset of the positive reals.

Y has more than one point because it is isomorphic to U, so by what we have just shown there exist $\beta, \gamma \epsilon B$ such that $0 < \beta < 1 < \gamma$. By induction on the fact that $y \epsilon Y$ and $\beta \epsilon B$ imply that $y\beta \epsilon Y$, we see that $y\beta^n$ and $y\gamma^n \epsilon Y$. Thus, when Y is a subset of the positive reals and when $x > 0$, we can choose n large enough so that $y\beta^n < x < y\gamma^n$. This, together with the fact that Y is an interval, implies that Y = positive reals. A similar argument holds when Y is a subset of the negative reals.

Suppose Y is the positive reals. Then $1 \epsilon Y$, and therefore

$$B \supseteq \{y/1 \mid y \epsilon Y\} = Y.$$

But B is a subset of the positive reals, so $B = Y$. If Y is the negative reals, a similar argument holds using $-1 \epsilon Y$.

If $\alpha \epsilon A$, then it is easy to show that $\alpha^{-1} = g^{-1}[1/g(\alpha)]$ gives the inverse operator. ||

COROLLARY 2. *Let (U, A, F) be a complete quasi-multiplicative family of stochastic operators that satisfies the hypothesis of Theorem 9. If there exist real-valued functions a, b, c, and d on A such that for all $p \in U$ and $\alpha \in A$,*

$$F(p, \alpha) = \frac{a(\alpha)p + b(\alpha)}{c(\alpha)p + d(\alpha)},$$

then either

i. *for all $\alpha \in A$, $b(\alpha) = 0$ and $c(\alpha) = a(\alpha) - d(\alpha)$, which is the family of beta model operators, or*

ii. *for all $\alpha \in A$, either $c(\alpha) = 0$ and $b(\alpha) = [d(\alpha) - a(\alpha)]/2$ or $c(\alpha) = 2a(\alpha)$ and $d(\alpha) = 2b(\alpha)$, which together form the family of linear operators with $\lambda = \frac{1}{2}$.*

PROOF. By the assumption that the family is quasi-multiplicative, there exist functions f and g such that

$$f\left[\frac{a(\alpha)p + b(\alpha)}{c(\alpha)p + d(\alpha)}\right] = f(p)g(\alpha),$$

and by Theorem 9 either Eqs. (10) and (11) hold or Eq. (12) holds.

Suppose that $d(\alpha) = 0$. Then $F(p, \alpha) = a(\alpha)/c(\alpha) + b(\alpha)/c(\alpha)p$. If $b(\alpha) \neq 0$, we can choose p sufficiently near 0 so that this quantity does not lie in the unit interval, which is impossible. Thus, $b(\alpha) = 0$. Substituting this into the above equation, we obtain

$$f[a(\alpha)/c(\alpha)] = f(p)g(\alpha),$$

which is possible only if $g(\alpha) = 0$. Because this result is inconsistent with Eq. (11), Eq. (12) must hold. Thus, $f(\frac{1}{2}) = 0$, and by the strict monotonicity of f, $c(\alpha) = 2a(\alpha)$. Therefore the case $d(\alpha) = 0$ falls under the second part of ii. In the rest of the argument, we assume that $d(\alpha) \neq 0$.

i. Suppose that f satisfies Eqs. (10) and (11). By Corollary 1, the range of f is either the positive or the negative reals, which with the strict monotonicity of f means either that

$$\lim_{p\to 0} f(p) = 0 \quad \text{and} \quad \lim_{p\to 1} 1/f(p) = 0$$

or that

$$\lim_{p\to 0} 1/f(p) = 0 \quad \text{and} \quad \lim_{p\to 1} f(p) = 0.$$

Assuming the first case, we obtain

$$\begin{aligned} 0 &= \lim_{p\to 0} f(p)g(\alpha) \\ &= \lim_{p\to 0} f\left[\frac{a(\alpha)p + b(\alpha)}{c(\alpha)p + d(\alpha)}\right] \\ &= f[b(\alpha)/d(\alpha)]. \end{aligned}$$

Thus, by the strict monotonicity of f, $b(\alpha) = 0$. Similarly,

$$0 = \lim_{p\to 1} 1/f(p)g(\alpha) = 1/f\left[\frac{a(\alpha) + b(\alpha)}{c(\alpha) + d(\alpha)}\right].$$

So, using $b(\alpha) = 0$, we obtain $c(\alpha) = a(\alpha) - d(\alpha)$.

A parallel argument yields the same conclusions when the range of f is the negative reals.

Substituting, we see that

$$F(p, \alpha) = \frac{[a(\alpha)/d(\alpha)]p}{[a(\alpha)/d(\alpha) - 1]p + 1},$$

which is a beta-model operator.

ii. If f satisfies Eq. (12), then

$$f(\tfrac{1}{2}) = 0 = f(\tfrac{1}{2})g(\alpha) = f\left[\frac{a(\alpha)/2 + b(\alpha)}{c(\alpha)/2 + d(\alpha)}\right],$$

so by the strict monotonicity of f,

$$b(\alpha) + [a(\alpha) - d(\alpha)]/2 = c(\alpha)/4.$$

By Eq. (12),

$$\begin{aligned} f\left[\frac{a(\alpha)p + b(\alpha)}{c(\alpha)p + d(\alpha)}\right] &= f(p)g(\alpha) \\ &= -f(1-p)g(\alpha) \\ &= -f\left[\frac{a(\alpha)(1-p) + b(\alpha)}{c(\alpha)(1-p) + d(\alpha)}\right]. \end{aligned}$$

The use of Eq. (12) and of the strict monotonicity of f implies that

$$\frac{a(\alpha)(1-p) + b(\alpha)}{c(\alpha)(1-p) + d(\alpha)} = 1 - \frac{a(\alpha)p + b(\alpha)}{c(\alpha)p + d(\alpha)}.$$

Cross multiplying and collecting terms, we obtain

$$c(\alpha)[c(\alpha) - 2a(\alpha)]p(1-p) + c(\alpha)[d(\alpha) - b(\alpha)] + d(\alpha)[d(\alpha) - a(\alpha) - 2b(\alpha)] = 0.$$

Because $d(\alpha) - a(\alpha) - 2b(\alpha) = -c(\alpha)/2$, the constant term can be reduced to $c(\alpha)[d(\alpha)/2 - b(\alpha)]$. For this equation to hold for all $p \in U$, both coefficients must be 0, so either $c(\alpha) = 0$ or $c(\alpha) = 2a(\alpha)$ and $d(\alpha) = 2b(\alpha)$. In the latter case,

$$F(p, \alpha) = \frac{a(\alpha)p + b(\alpha)}{2a(\alpha)p + 2b(\alpha)} = \tfrac{1}{2},$$

which is a special case of the linear operators with $\lambda = \frac{1}{2}$. In the former case, $b(\alpha) = [d(\alpha) - a(\alpha)]/2$, and so

$$F(p, \alpha) = [a(\alpha)/d(\alpha)]p + [1 - a(\alpha)/d(\alpha)]\tfrac{1}{2},$$

which is also a linear operator with $\lambda = \frac{1}{2}$. ‖

This corollary is not without interest. Let us suppose that the value of $F(pq, \alpha)$ is uniquely determined by the values of $F(p, \alpha)$ and $F(q, \alpha)$, or if we set $x = -\log p$, $y = -\log q$, and $G(x, \alpha) = F(e^{-x}, \alpha)$, that $G(x + y, \alpha)$ is a function of $G(x, \alpha)$ and $G(y, \alpha)$. If that function is rational, i.e., the ratio of two polynomials, then it is known (Aczél, 1961, p. 61) that there are only two possibilities for G:

$$G(x, \alpha) = \frac{a(\alpha)e^{-h(\alpha)x} + b(\alpha)}{c(\alpha)e^{-h(\alpha)x} + d(\alpha)} \quad \text{or} \quad G(x, \alpha) = \frac{a'(\alpha)x + b'(d)}{c'(\alpha)x + d'(\alpha)}.$$

Thus,

$$F(p,\alpha)=\frac{a(\alpha)p^{h(\alpha)}+b(\alpha)}{c(\alpha)p^{h(\alpha)}+d(\alpha)} \quad\text{or}\quad F(p,\alpha)=\frac{-a'(\alpha)\log p+b'(\alpha)}{-c'(\alpha)\log p+d'(\alpha)}.$$

It is easy to see that the first family is closed if and only if $h(\alpha)=1$ for all $\alpha \in A$ and that the second family cannot be closed. Thus, Corollary 2 applies and so allows us to conclude:

If (U, A, F) is a complete, quasi-multiplicative family of stochastic operators that satisfies the hypothesis of Theorem 9, and if, for all p, $q \in U$ and $\alpha \in A$, $F(pq, \alpha)$ is a rational function of $F(p, \alpha)$ and $F(q, \alpha)$, then the family is either the beta-model operators or the linear operators with $\lambda = \frac{1}{2}$.

6. Generalization of beta-model theorems

Kanal (1960, 1962a) has investigated a number of properties of the linear and beta models for the one-absorbing barrier (or 100 : 0 reward) situation. Both these models are special cases of quasi-multiplicative operators for which the range of f, Y, is a positive real interval beginning at 0. The transitions are given by

$$(13)\qquad y_{n+1}=\begin{cases}\beta_1 y_n & \text{with probability } p_n=f^{-1}(y_n),\\ \beta_2 y_n & \text{with probability } 1-p_n=1-f^{-1}(y_n),\end{cases}$$

where $0<\beta_1,\beta_2<1$ and $y_n>0$.

Adapting the methods of Tatsuoka and Mosteller (1959), Kanal showed that a number of interesting properties of these two stochastic processes—for example, the expected number of errors, all higher moments of errors, a weighted mean of errors, the number of trials before the first success, the trial of the last error, and various run statistics—satisfy functional equations of the form

$$(14)\ \phi(y,\beta_1,\beta_2)=f^{-1}(y)\phi(\beta_1 y,\beta_1,\beta_2)+[1-f^{-1}(y)]\phi(\beta_2 y,\beta_1,\beta_2) + h(y,\beta_1,\beta_2),$$

where $\phi(y,\beta_1,\beta_2)$ denotes the unknown value of the property when the stochastic process begins at $y_0=y$ and has parameters β_1 and β_2. His argument is independent of the form of the function f but depends upon commutativity, and so it applies to any quasi-multiplicative operators satisfying Eq. (13). In Eq. (14), f^{-1} depends upon the particular operators assumed, and h is a known function that depends both upon f and upon the property under investigation. In what follows, we will suppress the explicit dependence upon β_1 and β_2.

The results in Kanal's papers are worked out separately for

$$f^{-1}(y)=y/(1+y)\quad\text{and}\quad f^{-1}(y)=y,$$

but an examination of his proofs shows that they do not depend upon the form of f. For example, by paralleling the argument in Kanal (1960) we can prove:

Theorem 10. *For any stochastic process of the form of Eq. (13), let T be defined by*

$$T\psi(y)=f^{-1}(y)\psi(\beta_1 y)+[1-f^{-1}(y)]\psi(\beta_2 y)+h(y).$$

Then a solution to Eq. (14) is

$$\phi(y)=\lim_{n\to\infty}T^{(n)}h(y),$$

provided that the limit exists.

Proof. We define two sequences of functions, θ_n and ϕ_n, recursively as follows:

$$\theta_0(y)=y,\qquad \phi_0(y)=y,$$

and for $n\geq 1$,

$$\theta_{n+1}(y)=f^{-1}(y)\theta_n(\beta_1 y)+[1-f^{-1}(y)]\theta_n(\beta_2 y),$$
$$\phi_{n+1}(y)=\theta_{n+1}(y)+T^{(n)}h(y).$$

Observe that

$$\begin{aligned}\phi_{n+1}(y)&=\theta_{n+1}(y)+T^{(n)}h(y)\\&=f^{-1}(y)\theta_n(\beta_1 y)+[1-f^{-1}(y)]\theta_n(\beta_2 y)\\&\quad+f^{-1}(y)T^{(n-1)}h(\beta_1 y)+[1-f^{-1}(y)]T^{(n-1)}h(\beta_2 y)+h(y)\\&=f^{-1}(y)\phi_n(\beta_1 y)+[1-f^{-1}(y)]\phi_n(\beta_2 y)+h(y).\end{aligned}$$

Thus, $\phi(y)=\lim_{n\to\infty}\phi_n(y)$ is a solution to Eq. (14), provided that the limit exists.

To complete the proof, it is therefore sufficient to show that $\lim_{n\to\infty}\theta_n(y)=0$. First, we show that $\theta_n(\beta_i y)\leq\beta\theta_n(y)$, where

$$\beta=\max(\beta_1,\beta_2).$$

Note, $\theta_0(\beta_i y)=\beta_i v\leq\beta\theta_0(y)$. By induction,

$$\begin{aligned}\theta_{n+1}(\beta_i y)&=f^{-1}(y)\theta_n(\beta_1\beta_i y)+[1-f^{-1}(y)]\theta_n(\beta_2\beta_i y)\\&\leq f^{-1}(y)\beta\theta_n(\beta_1 y)+[1-f^{-1}(y)]\beta\theta_n(\beta_2 y)\\&=\beta\theta_{n+1}(y).\end{aligned}$$

Thus,

$$\begin{aligned}\theta_{n+1}(y)&=f^{-1}(y)\theta_n(\beta_1 y)+[1-f^{-1}(y)]\theta_n(\beta_2 y)\\&\leq f^{-1}(y)\beta\theta_n(y)+[1-f^{-1}(y)]\beta\theta_n(y)\\&=\beta\theta_n(y).\end{aligned}$$

Because $\beta<1$ and $\theta_n(y)\geq 0$ (by induction on n and the fact that $y>0$), it follows that the $\lim_{n\to\infty}\theta_n(y)=0$. ‖

By paralleling another of Kanal's proofs, one can show

Theorem 11. *If h is a strictly monotonic-increasing function and if $\Sigma_{i=0}^{\infty}h(\beta^i y)$ is finite for all $y \in Y$, where $\beta=\max(\beta_1,\beta_2)$, then $\lim_{n\to\infty}T^{(n)}h(y)$ exists.*

Moreover, an infinite series expression for a solution to Eq. (14) can be obtained that is formally the same as Kanal's Theorems 6 and 7. See Tatsuoka and Mosteller (1959) for more explicit power series for some properties of the linear model.

In contrast to the one-absorbing-barrier case, the results of Bush (1960) and Kanal (1962b) for the two-absorbing-barrier beta model do not readily generalize; the proofs they have developed depend critically upon the assumption that $f(p) = p/(1 - p)$.

A careful examination of the asymptotic beta-model results obtained by Lamperti and Suppes (1960) for the four-operator case with contingent reinforcement shows that the results are valid for any commutative operators for which Y is the positive reals, f is strictly monotonic-increasing,

$$\lim_{y\to\infty} f^{-1}(y) = 1, \quad \text{and} \quad \lim_{y\to\infty} f^{-1}(y) = 0 .$$

Some of the asymptotic theorems in Luce (1959) for this same situation, in particular the most important one (Theorem 17, p. 116), do not make essential use of the beta-model transformation, but other of the theorems do.

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